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Energy Methods in Dynamics
Exercises and Solutions

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Springer

Chapter 1
Single oscillator

EXERCISE 1.1 Derive the equation of motion of a roller (mass m, radius r) hung on an unstretchable rope and a spring (see Fig. 1.1) with the help of

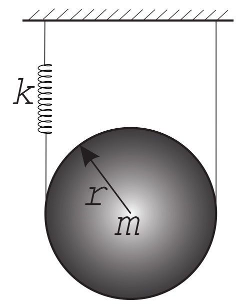

Fig. 1.1 Roller hung on rope and spring.

a. the force method,
b. the energy method.
Determine the eigenfrequency of vibration.

Solution. a) The force method. We free the roller from the rope and the spring (see Fig. 1.2) and apply the moment equation about A

$$\frac{d}{dt}(J_A\dot{\varphi}) = \sum M_z = -F_s 2r.$$

From the kinematics we know that $\varphi = x/2r$. Besides, the spring force is equal to $F_s = kx$, while the moment of inertia of the roller about A is

$$J_A = J_O + mr^2 = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2.$$

Thus, the equation of motion reads

$$\frac{3}{2}mr^2\frac{\ddot{x}}{2r} = -kx2r \quad \Rightarrow \quad \ddot{x} + \frac{8k}{3m}x = 0,$$

and the eigenfrequency is given by

$$\omega^2 = \frac{8k}{3m} \quad \Rightarrow \quad \omega = \sqrt{\frac{8k}{3m}}.$$

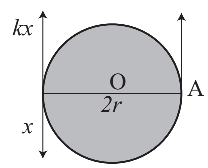

Fig. 1.2 Roller and the forces.

b) The energy method. We write down the kinetic and potential energies:

$$K = \frac{1}{2}J_A\dot{\varphi}^2, \quad U = \frac{1}{2}kx^2.$$

Taking into account the kinematic relation $\varphi = x/2r$ and the formula $J_A = \frac{3}{2}mr^2$ we obtain the Lagrange function in the form

$$L = K - U = \frac{3}{16}m\dot{x}^2 - \frac{1}{2}kx^2.$$

Then, from the Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0,$$

we derive again the above equation of motion and the formula for ω.

EXERCISE 1.2 Derive the equation of motion of a thin circular ring (mass m, radius r) hung on a support O (see Fig. 1.3). Determine the eigenfrequency of small vibration.

Solution. We write down the kinetic and potential energies:

$$K = \frac{1}{2}J_O\dot{\varphi}^2, \quad U = mgr(1 - \cos\varphi).$$

The moment of inertia of the thin ring is equal to

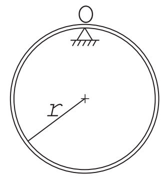

Fig. 1.3 Ring hung on support.

$$J_O = J_S + mr^2 = mr^2 + mr^2 = 2mr^2.$$

For small vibrations with $\varphi \ll 1$ we may approximate $1-\cos\varphi \approx \frac{1}{2}\varphi^2$. Thus,

$$L = mr^2\dot{\varphi}^2 - \frac{1}{2}mgr\varphi^2.$$

From Lagrange's equation we derive the equation of motion

$$2mr^2\ddot{\varphi} + mgr\varphi = 0.$$

Consequently, the eigenfrequency is

$$\omega = \sqrt{\frac{g}{2r}}.$$

EXERCISE 1.3 Three turning points are measured from the vibration of a damped oscillator: $x_1 = 8.6$mm, $x_2 = -4.1$mm, $x_3 = 4.3$mm. Determine the middle point of vibration (position of equilibrium). Find the logarithmic decrement ϑ and the damping ratio δ.

Solution. The free vibration of the underdamped oscillator is described by

$$x(\tau) = x_m + a_0 e^{-\delta\tau}\cos(\nu\tau - \phi),$$

where x_m corresponds to the position of equilibrium. As we know, $x(\tau)$ achieves maxima or minima if

$$\tan(\nu\tau - \phi) = -\delta/\nu,$$

so, the turning points (corresponding to maxima or minima) occur at the time instants τ_1, $\tau_1 + \tau_c/2$, $\tau_1 + \tau_c$ and so on, where $\tau_c = 2\pi/\nu$ is the conditional period of vibration. Taking the periodicity of cosine function into account, we have

$$x_1 = x_m + C,$$
$$x_2 = x_m - Ce^{-\delta\pi/\nu},$$
$$x_3 = x_m + Ce^{-\delta 2\pi/\nu},$$

where $C = a_0 e^{-\delta\tau_1}\cos(\nu\tau_1 - \phi)$, with τ_1 being the time instant of the first turning point. Forming the differences we easily see that

$$\frac{x_1 - x_2}{x_3 - x_2} = e^{\delta\pi/\nu} = e^{\delta\tau_c/2}.$$

Thus,

$$\ln\frac{x_1 - x_2}{x_3 - x_2} = \delta\tau_c/2 = \frac{\vartheta}{2},$$

with ϑ the logarithmic decrement. Substituting the given values of turning points into this formula we obtain

$$\vartheta = 2\ln\frac{x_1 - x_2}{x_3 - x_2} = 0.827.$$

Knowing the logarithmic decrement, we find Lehr's damping ratio

$$\delta = \frac{\vartheta}{\sqrt{4\pi^2 + \vartheta^2}} = 0.13.$$

Now we form the differences $x_1 - x_m$ and $x_m - x_2$ and consider the quotient

$$\frac{x_1 - x_m}{x_m - x_2} = e^{\delta\tau_c/2} = e^{\vartheta/2}.$$

From the last equation we find x_m

$$x_m = \frac{x_1 + x_2 e^{\vartheta/2}}{1 + e^{\vartheta/2}} = 0.956\text{mm}.$$

EXERCISE 1.4 The time constants are measured from the vibration of a damped oscillator: $T_d = 5$s, $T_c = 2$s. Determine ϑ and δ.

EXERCISE 1.5 Determine the unit step responses for the overdamped and the critically damped oscillator.

EXERCISE 1.6 Find the solution of the initial-value problem

$$x'' + 2\delta x' + x = 0,$$

satisfying $x(0) = x_0$ and $x'(0) = x'_0$ with the help of the Laplace transform.

EXERCISE 1.7 Use Duhamel's formula to compute the response of the damped oscillator with $\delta = 1$ to the so-called ramp function

$$g(\tau) = \begin{cases} 0 & \text{for } \tau \le 0, \\ \alpha\tau & \text{for } 0 \le \tau \le \tau_0, \\ \alpha\tau_0 & \text{for } \tau \ge \tau_0. \end{cases}$$

The oscillator was at rest for $\tau \leq 0$.

Solution. According to Duhamel's formula

$$x(\tau) = \int_0^{\tau} g'(t) x_r(\tau - t)\, dt,$$

where $x_r(\tau)$ is the unit step response function. For $\delta = 1$ we have

$$x_r(\tau) = 1 - (1+\tau)e^{-\tau}.$$

We compute the derivative of the ramp function $g(\tau)$ given above

$$g'(\tau) = \begin{cases} 0 & \text{for } \tau < 0, \\ \alpha & \text{for } 0 \leq \tau \leq \tau_0, \\ 0 & \text{for } \tau > \tau_0. \end{cases}$$

So we need to consider two different cases.

Case a: $\tau < \tau_0$. In this case

$$\begin{aligned} x(\tau) &= \int_0^{\tau} \alpha[1 - (1 + (\tau - t))e^{-(\tau - t)}]\, dt \\ &= \alpha \left[\tau - \int_0^{\tau} e^{-(\tau - t)}\, dt - \int_0^{\tau} (\tau - t) e^{-(\tau - t)}\, dt \right] \\ &= \alpha \left[\tau - \int_{-\tau}^{0} e^{u}\, du - \int_{-\tau}^{0} u e^{u}\, du \right] \\ &= \alpha(\tau - 2 + 2e^{-\tau} + \tau e^{-\tau}). \end{aligned}$$

Case b: $\tau > \tau_0$. Since $g'(\tau) = 0$ for $\tau > \tau_0$ we have

$$\begin{aligned} x(\tau) &= \int_0^{\tau_0} \alpha[1 - (1 + (\tau - t))e^{-(\tau - t)}]\, dt \\ &= \alpha \left[\tau_0 - \int_0^{\tau_0} e^{-(\tau - t)}\, dt - \int_0^{\tau_0} (\tau - t) e^{-(\tau - t)}\, dt \right] \\ &= \alpha \left[\tau_0 - \int_{-\tau}^{\tau_0 - \tau} e^{u}\, du - \int_{-\tau}^{\tau_0 - \tau} u e^{u}\, du \right] \\ &= \alpha[\tau_0 - (2 + \tau - \tau_0)e^{-(\tau - \tau_0)} + (\tau + 2)e^{-\tau}]. \end{aligned}$$

EXERCISE 1.8 Derive the equations of motion in examples 1.8 and 1.9 by the energy method.

EXERCISE 1.9 Derive the equation of vertical motion of a frame (mass M) excited by two rotating unbalanced masses (mass $m/2$, frequency of rotation ω, radius of rotation r). The frame is connected with two springs of equal stiffness $k/2$ and a damper with damping constant c (see Fig. 1.4). Determine the magnification factor of forced vibration.

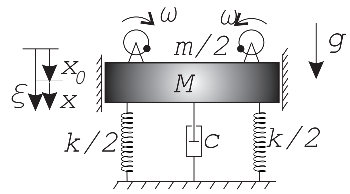

Fig. 1.4 Vertical forced vibration of frame.

Solution. We derive the equation of motion by the energy method. Let x be the vertical displacement of the center of mass of the frame from the equilibrium position. The displacements of the unbalanced masses are described by

$$x_u = x + r\cos\omega t, \quad y_u = \pm r\sin\omega t.$$

The velocities of the unbalanced masses are

$$\dot{x}_u = \dot{x} - r\omega\sin\omega t, \quad \dot{y}_u = \pm r\omega\cos\omega t.$$

Thus, the kinetic energy of masses equals

$$K(\dot{x}) = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x} - r\omega\sin\omega t)^2 + \frac{1}{2}mr^2\omega^2\cos^2\omega t.$$

Since the gravitational force and the static spring forces do not contribute to the potential energy, we have

$$U(x) = \frac{1}{2}kx^2.$$

The Lagrange function reads

$$L = K - U = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x} - r\omega\sin\omega t)^2 + \frac{1}{2}mr^2\omega^2\cos^2\omega t - \frac{1}{2}kx^2.$$

The dissipation function is given by

$$D(\dot{x}) = \frac{1}{2}c\dot{x}^2.$$

From the modified Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = 0$$

we derive the equation of motion of this system

$$(M+m)\ddot{x} + c\dot{x} + kx = mr\omega^2\cos\omega t.$$

The eigenfrequency of free vibration is $\omega_0 = \sqrt{\frac{k}{M+m}}$. Introducing the dimensionless time $\tau = \omega_0 t$, we reduce the equation of motion to the standard form

$$x'' + 2\delta x' + x = x_0 \eta^2 \cos \eta \tau,$$

where

$$\delta = \frac{c\omega_0}{2k}, \quad x_0 = \frac{m}{M+m} r, \quad \eta = \frac{\omega}{\omega_0}.$$

The forced vibration reads

$$x = x_0 M \cos(\eta \tau - \psi) = x_0 M (\cos \eta \tau \cos \psi + \sin \eta \tau \sin \psi),$$

where M is called a magnification factor and ψ the phase of forced vibration. The magnification factor is equal to (see Section 1.4)

$$M = \frac{\eta^2}{\sqrt{(1-\eta^2)^2 + 4\delta^2 \eta^2}}.$$

EXERCISE 1.10 Show that the variational problem

$$\delta \int_0^{2\pi} \left(\frac{1}{2} x'^2 - \frac{1}{2} x^2 + \cos \tau x\right) d\tau = 0$$

has no extremal in the class of periodic functions with $x(0) = x(2\pi)$ and $x'(0) = x'(2\pi)$. Find its extremal. What happens if the last term in the integrand is $\sin \tau x$.

EXERCISE 1.11 Find the maxima of the magnification factors M in three cases a, b, and c considered in Section 1.4.

EXERCISE 1.12 Find the idle and active works done by the external force in cases b and c considered in Section 1.4.

Chapter 2
Coupled oscillators

EXERCISE 2.1 Two point-masses m_1 and m_2 are connected with a fixed support O and with each other by two rigid and massless bars of lengths l_1 and l_2 (see Fig. 2.1). Derive the equations of small vibration of this double pendulum under the action of gravity. Determine the eigenfrequencies of vibrations.

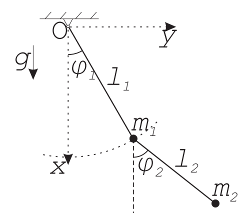

Fig. 2.1 Double pendulum.

Solution. This system has two degrees of freedom described by the angles φ_1 and φ_2. Let us write down the kinetic and potential energies of this double pendulum. For the kinetic energy we have

$$K(\dot{\varphi}_i) = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2.$$

As the first point-mass m_1 rotates about O with the angular velocity $\dot{\varphi}_1$, the magnitude of its velocity is $v_1 = l_1\dot{\varphi}_1$. The velocity of m_2 is the superposition of the velocity of m_1 and the relative velocity of m_2 with respect to m_1, so

$$\mathbf{v}_2 = \mathbf{v}_1 + \mathbf{v}_{21}.$$

Since both angles φ_1 and φ_2 are small, these two vectors are nearly parallel. Taking into account that $v_{21} = l_2\dot{\varphi}_2$, we can write

$$v_2^2 = v_1^2 + v_{21}^2 + 2\mathbf{v}_1\cdot\mathbf{v}_{21} = l_1^2\dot{\varphi}_1^2 + l_2^2\dot{\varphi}_2^2 + 2l_1 l_2\dot{\varphi}_1\dot{\varphi}_2.$$

Thus, the kinetic energy is equal to

$$K = \frac{1}{2}(m_1+m_2)l_1^2\dot{\varphi}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\varphi}_2^2 + m_2 l_1 l_2 \dot{\varphi}_1\dot{\varphi}_2.$$

The potential energy of the point-masses in the gravitational field is given by

$$U = m_1 g l_1(1-\cos\varphi_1) + m_2 g(l_1+l_2-l_1\cos\varphi_1 - l_2\cos\varphi_2).$$

For small angles φ_1 and φ_2 this can be approximated by

$$U = \frac{1}{2}m_1 g l_1\varphi_1^2 + \frac{1}{2}m_2 g l_1\varphi_1^2 + \frac{1}{2}m_2 g l_2\varphi_2^2.$$

Thus, the Lagrange function reads

$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot{\varphi}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\varphi}_2^2 + m_2 l_1 l_2\dot{\varphi}_1\dot{\varphi}_2 - \frac{1}{2}(m_1+m_2)g l_1\varphi_1^2 - \frac{1}{2}m_2 g l_2\varphi_2^2.$$

From the Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}_j} - \frac{\partial L}{\partial\varphi_j} = 0, \quad j = 1,2$$

we derive the equations of motion

$$\begin{aligned}\frac{d}{dt}((m_1+m_2)l_1^2\dot{\varphi}_1 + m_2 l_1 l_2\dot{\varphi}_2) + (m_1+m_2)g l_1\varphi_1 = 0,\\ \frac{d}{dt}(m_2 l_2^2\dot{\varphi}_2 + m_2 l_1 l_2\dot{\varphi}_1) + m_2 g l_2\varphi_2 = 0.\end{aligned}$$

Dividing the first equation by l_1 and the second one by $m_2 l_2$, we reduce this system to

$$\begin{aligned}(m_1+m_2)l_1\ddot{\varphi}_1 + m_2 l_2\ddot{\varphi}_2 + (m_1+m_2)g\varphi_1 = 0,\\ l_1\ddot{\varphi}_1 + l_2\ddot{\varphi}_2 + g\varphi_2 = 0.\end{aligned}$$

To determine the eigenfrequencies of vibrations we seek for the solution in the form

$$\varphi_j = \hat{\varphi}_j e^{i\omega t}.$$

Substituting this into the equations of motion we get

$$\begin{pmatrix}(m_1+m_2)(g-l_1\omega^2) & -m_2 l_2\omega^2\\ -l_1\omega^2 & g-l_2\omega^2\end{pmatrix}\begin{pmatrix}\hat{\varphi}_1\\ \hat{\varphi}_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Non-trivial solutions of this equation exist if its determinant vanishes

$$\begin{vmatrix}(m_1+m_2)(g-l_1\omega^2) & -m_2 l_2\omega^2\\ -l_1\omega^2 & g-l_2\omega^2\end{vmatrix} = 0.$$

Computing the determinant, we get the following characteristic equation

$$m_1 l_1 l_2 \omega^4 - (m_1+m_2)g(l_1+l_2)\omega^2 + (m_1+m_2)g^2 = 0.$$

Solving this quadratic equation (with respect to ω^2) we obtain two roots

$$\omega_{1,2}^2 = \frac{g}{2m_1 l_1 l_2}\left[(m_1+m_2)(l_1+l_2) \mp \sqrt{(m_1+m_2)[(m_1+m_2)(l_1+l_2)^2 - 4m_1 l_1 l_2]}\right].$$

EXERCISE 2.2 A body of mass m is connected with the wall through a spring of stiffness k and with a bar of length l and equal mass m which rotates in the plane about S (see Fig. 2.2). Derive the equations of small vibration of this system. Determine the eigenfrequencies of vibrations.

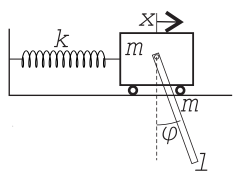

Fig. 2.2 Body connected with spring and bar.

Solution. Let $\mathbf{q} = (x, \varphi)$ be the generalized coordinates. We write down the kinetic energy of this system

$$K(\dot{\mathbf{q}}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mv_S^2 + \frac{1}{2}J_S\dot{\varphi}^2,$$

where the last two terms represent the kinetic energy of the bar, with v_S the velocity of the center of mass and $J_S = ml^2/12$ the moment of inertia of the bar about S. For small angle $\varphi \ll 1$

$$v_S = \dot{x} + \frac{l}{2}\dot{\varphi}.$$

So, the kinetic energy of this system reads

$$K(\dot{\mathbf{q}}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(\dot{x} + \frac{l}{2}\dot{\varphi})^2 + \frac{1}{24}ml^2\dot{\varphi}^2.$$

Concerning the potential energy we have for small angle

$$U(\mathbf{q}) = \frac{1}{2}kx^2 + mg\frac{l}{2}(1-\cos\varphi) \approx \frac{1}{2}kx^2 + mg\frac{l}{4}\varphi^2.$$

Thus,

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(\dot{x} + \frac{l}{2}\dot{\varphi})^2 + \frac{1}{24}ml^2\dot{\varphi}^2 - \frac{1}{2}kx^2 - mg\frac{l}{4}\varphi^2.$$

From the Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad j = 1,2$$

we derive the equations of motion

$$m\ddot{x} + m(\ddot{x} + \frac{l}{2}\ddot{\varphi}) + kx = 0,$$
$$m\frac{l}{2}(\ddot{x} + \frac{l}{2}\ddot{\varphi}) + \frac{1}{12}ml^2\ddot{\varphi} + \frac{1}{2}mgl\varphi = 0.$$

These equations can be simplified to

$$2m\ddot{x} + m\frac{l}{2}\ddot{\varphi} + kx = 0,$$
$$\frac{1}{3}ml^2\ddot{\varphi} + m\frac{l}{2}\ddot{x} + \frac{1}{2}mgl\varphi = 0.$$

Dividing the first equation by $2m$ and the second one by $ml/2$, respectively, we rewrite them in the form

$$\ddot{x} + \frac{l}{4}\ddot{\varphi} + \omega_x^2 x = 0,$$
$$\frac{3}{2l}\ddot{x} + \ddot{\varphi} + \omega_\varphi^2 \varphi = 0,$$

where

$$\omega_x^2 = \frac{k}{2m}, \quad \omega_\varphi^2 = \frac{3g}{2l}.$$

To determine the eigenfrequencies of vibrations we seek for the solution in the form

$$\begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{\varphi} \end{pmatrix} e^{i\omega t}.$$

Substituting this into the equations of motion we get

$$\begin{pmatrix} -\omega^2 + \omega_x^2 & -\frac{l}{4}\omega^2 \\ -\frac{3}{2l}\omega^2 & -\omega^2 + \omega_\varphi^2 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Non-trivial solutions of this equation exist if its determinant vanishes

$$\begin{vmatrix} -\omega^2 + \omega_x^2 & -\frac{l}{4}\omega^2 \\ -\frac{3}{2l}\omega^2 & -\omega^2 + \omega_\varphi^2 \end{vmatrix} = 0.$$

Computing the determinant, we get the following characteristic equation

$$(-\omega^2 + \omega_x^2)(-\omega^2 + \omega_\varphi^2) - \frac{3}{8}\omega^4 = 0,$$

yielding two roots

$$\omega_{1,2}^2 = \frac{4}{5}\left(\omega_x^2 + \omega_\varphi^2 \mp \sqrt{(\omega_x^2 + \omega_\varphi^2)^2 - \frac{5}{2}\omega_x^2\omega_\varphi^2}\right).$$

EXERCISE 2.3 A rigid bar of mass m and moment of inertia $J_S = m\rho^2$ is hung on two massless and unstretchable robes of equal length l (this is the primitive mechanical model of the swing). The distance between the robes in the equilibrium state is s. The distances between the attachment points and the center of mass of the bar are s_1 and s_2, respectively. Under the assumption $\varphi_1 \ll 1$, $\varphi_2 \ll 1$ derive the equations of out-of-plane vibration of the bar, neglecting its in-plane motion. Determine the eigenfrequencies of vibrations.

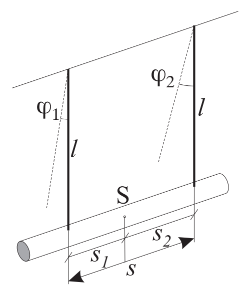

Fig. 2.3 Bar hung on two robes.

Solution. The motion of the bar as rigid body is the superposition of the translation of the center of mass S and the rotation about S. Accordingly, the kinetic energy of the bar equals

$$K = \frac{1}{2}mv_S^2 + \frac{1}{2}J_S\omega^2,$$

where ω is the angular velocity and J_S the moment of inertia of the bar about S. This motion can also be regarded as the pure rotation about the instantaneous center of rotation P with the same angular velocity ω (see Fig. 2.4).

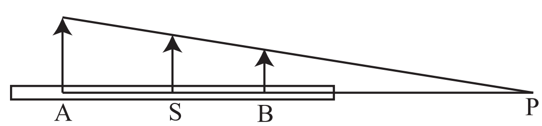

Fig. 2.4 Pure rotation of the bar about P.

The velocities of the attachment points A and B are $l\dot{\varphi}_1$ and $l\dot{\varphi}_2$, respectively. Let the distance between A and P be x, then the distance between B and P is $x - s_1 - s_2$, so

$$x\omega = l\dot{\varphi}_1$$
$$(x - s_1 - s_2)\omega = l\dot{\varphi}_2.$$

From here we find that

$$\omega = \frac{l}{s_1 + s_2}(\dot{\varphi}_1 - \dot{\varphi}_2), \quad x = \frac{(s_1 + s_2)\dot{\varphi}_1}{\dot{\varphi}_1 - \dot{\varphi}_2}$$

The velocity of the center of mass, v_S, can also be easily found as

$$v_S = (x - s_1)\omega = l\left(\frac{s_2}{s_1 + s_2}\dot{\varphi}_1 + \frac{s_1}{s_1 + s_2}\dot{\varphi}_2\right).$$

Thus, the kinetic energy of the bar reads

$$K = \frac{1}{2}ml^2\left(\frac{s_2}{s_1 + s_2}\dot{\varphi}_1 + \frac{s_1}{s_1 + s_2}\dot{\varphi}_2\right)^2 + \frac{1}{2}m\rho^2\frac{l^2}{(s_1 + s_2)^2}(\dot{\varphi}_1 - \dot{\varphi}_2)^2.$$

To write down the potential energy of the bar we find out the change of height of the center of mass. The change of height of the attachment points A and B are

$$w_1 = l(1 - \cos\varphi_1) \approx l\frac{\varphi_1^2}{2}, \quad w_2 = l(1 - \cos\varphi_2) \approx l\frac{\varphi_2^2}{2}.$$

For the bar, the change of height must be a linear function of x:

$$w(x) = ax + b,$$

where x is the coordinate along the bar axis. Choosing $x = 0$ at A, we find that $b = w_1$. For $x = s_1 + s_2$ at B we have $a(s_1 + s_2) + w_1 = w_2$, so

$$a = \frac{w_2 - w_1}{s_1 + s_2}.$$

Consequently, the change of height of the center of mass equals

$$w_S = \frac{l}{2(s_1 + s_2)}(s_2\varphi_1^2 + s_1\varphi_2^2),$$

and the potential energy reads

$$U = \frac{mgl}{2(s_1 + s_2)}(s_2\varphi_1^2 + s_1\varphi_2^2).$$

Combining the kinetic and potential energies, we obtain the Lagrange function in the form

$$L = \frac{ml^2}{2(s_1 + s_2)^2}(s_2\dot{\varphi}_1 + s_1\dot{\varphi}_2)^2 + \frac{m\rho^2 l^2}{2(s_1 + s_2)^2}(\dot{\varphi}_1 - \dot{\varphi}_2)^2 - \frac{mgl}{2(s_1 + s_2)}(s_2\varphi_1^2 + s_1\varphi_2^2).$$

The Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_j} - \frac{\partial L}{\partial \varphi_j} = 0, \quad j = 1,2$$

leads to

$$\frac{d}{dt}\left[\frac{ml^2 s_2}{(s_1+s_2)^2}(s_2\dot{\varphi}_1 + s_1\dot{\varphi}_2) + \frac{m\rho^2 l^2}{(s_1+s_2)^2}(\dot{\varphi}_1 - \dot{\varphi}_2)\right] + \frac{mgls_2}{s_1+s_2}\varphi_1 = 0,$$
$$\frac{d}{dt}\left[\frac{ml^2 s_1}{(s_1+s_2)^2}(s_2\dot{\varphi}_1 + s_1\dot{\varphi}_2) - \frac{m\rho^2 l^2}{(s_1+s_2)^2}(\dot{\varphi}_1 - \dot{\varphi}_2)\right] + \frac{mgls_1}{s_1+s_2}\varphi_2 = 0.$$

Dividing both equations by $ml^2/(s_1+s_2)^2$ we reduce them to

$$(s_2^2+\rho^2)\ddot{\varphi}_1 + (s_1 s_2 - \rho^2)\ddot{\varphi}_2 + \frac{gss_2}{l}\varphi_1 = 0,$$
$$(s_1 s_2 - \rho^2)\ddot{\varphi}_1 + (s_1^2+\rho^2)\ddot{\varphi}_2 + \frac{gss_1}{l}\varphi_2 = 0,$$

where $s = s_1 + s_2$.

To determine the eigenfrequencies of vibrations we seek for the solution in the form

$$\varphi_j = \hat{\varphi}_j e^{i\omega t}.$$

Substituting this into the equations of motion we get

$$\begin{pmatrix} (\frac{gss_2}{l} - (s_2^2+\rho^2)\omega^2) & -(s_1 s_2 - \rho^2)\omega^2 \\ -(s_1 s_2 - \rho^2)\omega^2 & (\frac{gss_1}{l} - (s_1^2+\rho^2)\omega^2) \end{pmatrix} \begin{pmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From the condition of vanishing determinant, we get the following characteristic equation

$$(\frac{gss_2}{l} - (s_2^2+\rho^2)\omega^2)(\frac{gss_1}{l} - (s_1^2+\rho^2)\omega^2) - (s_1 s_2 - \rho^2)^2\omega^4 = 0,$$

which can be reduced to

$$\rho^2\omega^4 - \frac{g}{l}(s_1 s_2 + \rho^2)\omega^2 + \frac{g^2}{l^2}s_1 s_2 = 0.$$

Solving this quadratic equation (with respect to ω^2) we obtain two roots

$$\omega_1^2 = \frac{g}{l}, \quad \omega_2^2 = \frac{gs_1 s_2}{l\rho^2}.$$

EXERCISE 2.4 Beating phenomenon. Find solution of (2.7) for the coupled pendulums satisfying the initial conditions: $\varphi_1(0) = 1$, $\varphi_2(0) = \dot{\varphi}_1(0) = \dot{\varphi}_1(0) = 0$. Plot $\varphi_1(t)$ and $\varphi_2(t)$ for $\alpha = 0.1$ and analyze their behavior.

EXERCISE 2.5 Consider a pair of uncoupled harmonic oscillators described by the equations $\ddot{x}+x=0$ and $\ddot{y}+\omega^2 y=0$. Using t as parameter, plot the trajectory of the motion in the (x,y)-plane given by $x(t)=\cos t$ and $y(t)=\cos\omega t$ for $t\in(0,1000)$ in two cases: i) $\omega=3$ and ii) $\omega=\pi$. The curves of this type are called Lissajous figures, and due to the periodicity in x and y the trajectories can be regarded as moving on a two-dimensional torus. Observe the difference in cases i) and ii).

EXERCISE 2.6 Determine the vibration modes and the normal coordinates of the double pendulum with $m_1=m_2=m$ and $l_1=l_2=l$.

Solution. Under the conditions $m_1=m_2=m$ and $l_1=l_2=l$ the Lagrange function, as seen from the solution of the exercise 2.1, is given by

$$L=\frac{1}{2}ml^2\dot{\varphi}_1^2+\frac{1}{2}ml^2(\dot{\varphi}_1+\dot{\varphi}_2)^2-mgl\varphi_1-\frac{1}{2}mgl\varphi_2.$$

The division of this Lagrange function by ml^2 does not influence the equations of motion, so we can write

$$L=\frac{1}{2}\dot{\varphi}_1^2+\frac{1}{2}(\dot{\varphi}_1+\dot{\varphi}_2)^2-\omega_0^2\varphi_1-\frac{1}{2}\omega_0^2\varphi_2,$$

where $\omega_0^2=g/l$. The equations of motion in the matrix form reads

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{0},$$

where

$$\mathbf{M}=\begin{pmatrix}2&1\\1&1\end{pmatrix},\quad \mathbf{K}=\begin{pmatrix}2\omega_0^2&0\\0&\omega_0^2\end{pmatrix}.$$

The problem is to bring both matrices to the diagonal form. This can be realized by solving the eigenvalue problem

$$(-\omega^2\mathbf{M}+\mathbf{K})\mathbf{q}=\mathbf{0},$$

or,

$$\begin{pmatrix}-2\omega^2+2\omega_0^2&-\omega^2\\-\omega^2&-\omega^2+\omega_0^2\end{pmatrix}\begin{pmatrix}\hat{q}_1\\\hat{q}_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

The characteristic equation

$$\det(-\omega^2\mathbf{M}+\mathbf{K})=2(\omega^2-\omega_0^2)^2-\omega^4=0$$

yields two eigenfrequencies

$$\omega_{1,2}^2=\omega_0^2(2\mp\sqrt{2}).$$

The corresponding eigenvectors of these two modes of vibrations are

$$\mathbf{q}_1 = \begin{pmatrix} 2-\sqrt{2} \\ 2(-1+\sqrt{2}) \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} -2-\sqrt{2} \\ 2(1+\sqrt{2}) \end{pmatrix}.$$

With these eigenvectors we can form the modal matrix

$$\mathbf{Q} = \begin{pmatrix} 2-\sqrt{2} & -2-\sqrt{2} \\ 2(-1+\sqrt{2}) & 2(1+\sqrt{2}) \end{pmatrix}. \tag{2.1}$$

Thus, the normal coordinates are

$$\begin{aligned} \xi_1 &= (2-\sqrt{2})\varphi_1 - (2+\sqrt{2})\varphi_2, \\ \xi_2 &= 2(-1+\sqrt{2})\varphi_1 + 2(1+\sqrt{2})\varphi_2. \end{aligned}$$

EXERCISE 2.7 Determine the vibration modes and the normal coordinates in exercise 2.3.

Solution. From the solution of exercise 2.3 we see that there are two eigenfrequencies of vibrations

$$\omega_1^2 = \frac{g}{l}, \quad \omega_2^2 = \frac{g s_1 s_2}{l\rho^2}.$$

Let us find out the corresponding eigenvectors. For mode 1 with $\omega_1^2 = \frac{g}{l}$ we have

$$\frac{g}{l}\begin{pmatrix} s_1 s_2 - \rho^2 & -(s_1 s_2 - \rho^2) \\ -(s_1 s_2 - \rho^2) & s_1 s_2 - \rho^2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Together with the normalization condition $\mathbf{q}_1 \cdot \mathbf{M}\mathbf{q}_1 = 1$ we find that

$$\mathbf{q}_1 = \begin{pmatrix} 1/s \\ 1/s \end{pmatrix}.$$

Thus, this mode of vibration corresponds to the synchronized parallel motion of the bar with $\varphi_1 = \varphi_2$ (the swing mode). For mode 2 with $\omega_2^2 = \frac{g s_1 s_2}{l\rho^2}$ we have

$$\frac{g}{l}\begin{pmatrix} s s_2 - (s_2^2+\rho^2)\frac{s_1 s_2}{\rho^2} & -(s_1 s_2 - \rho^2)\frac{s_1 s_2}{\rho^2} \\ -(s_1 s_2 - \rho^2)\frac{s_1 s_2}{\rho^2} & s s_1 - (s_2^1+\rho^2)\frac{s_1 s_2}{\rho^2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consequently,

$$\frac{q_2}{q_1} = \frac{s s_2 - (s_2^2+\rho^2)\frac{s_1 s_2}{\rho^2}}{(s_1 s_2 - \rho^2)\frac{s_1 s_2}{\rho^2}} = -\frac{s_2}{s_1}.$$

Together with the normalization condition $\mathbf{q}_2 \cdot \mathbf{M}\mathbf{q}_2 = 1$ we find that

$$\mathbf{q}_1 = \frac{1}{\rho}\begin{pmatrix} -s_1/(s_2 - s_1) \\ s_2/(s_2 - s_1) \end{pmatrix}.$$

This mode of vibration describes the rotation of the bar about the center of mass (antisymmetric mode). Thus, the modal matrix equals

$$\mathbf{Q} = \begin{pmatrix} 1/s & -s_1/(\rho(s_2 - s_1)) \\ 1/s & s_2/(\rho(s_2 - s_1)) \end{pmatrix}.$$

and the normal coordinates are

$$\xi_1 = \frac{1}{s}\varphi_1 - \frac{s_1}{\rho(s_2 - s_1)}\varphi_2,$$
$$\xi_2 = \frac{1}{s}\varphi_1 + \frac{s_2}{\rho(s_2 - s_1)}\varphi_2.$$

EXERCISE 2.8 Find the coordinates of the fixed points A and B of resonance curves in example 2.8. Show that A and B are at equal level when

$$\kappa = \frac{\mu}{(1+\mu)^2}.$$

EXERCISE 2.9 Find the solution of example 2.9 by Laplace's transform and show that it is equal to the solution found by the modal decomposition.

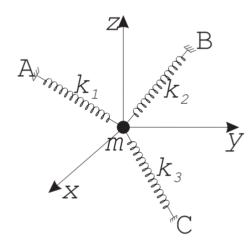

Fig. 2.5 Mass-spring oscillator with 3 degrees of freedom.

EXERCISE 2.10 A point-mass m moves in the space under the action of three springs of stiffnesses k_1, k_2, and k_3 the axes of which do not lie in one plane (see Fig. 2.5). The equilibrium position of the point-mass is chosen as the origin of the coordinate system, while $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ denote the unit vectors along the spring axes. Derive the equation of small vibrations for this oscillator and determine the eigenfrequencies.

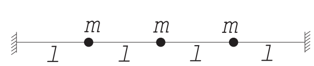

Fig. 2.6 Pre-stretched string with 3 point-masses.

EXERCISE 2.11 A pre-stretched string contains three equal and equally spaced point-masses m (see Fig. 2.6). The tension in the string is assumed to be large, so that for small lateral displacements of the point-masses it does not change appreciably. Derive the equation of small lateral vibration and determine the eigenfrequencies.

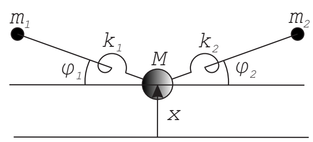

Fig. 2.7 A primitive model of an airplane with 3 degrees of freedom.

EXERCISE 2.12 The free vibrations of an airplane can be described in a simplified model with three degrees of freedom representing the motion of the fuselage and the wings which are connected with the fuselage by the spiral springs of stiffnesses k_1 and k_2 (see Fig. 2.7). Derive the equations of small vibrations. Under the assumptions of symmetry $\theta_1 = \theta_2 = \theta$, $m_1 = m_2 = m$, and $k_1 = k_2 = k$, find the eigenfrequencies of vibrations. Discuss the case when the symmetry assumption is removed.

Chapter 3
Continuous oscillators

EXERCISE 3.1 Derive the equation of motion for a chain of atoms, where each atom interacts with m neighbors on the left as well as m neighbors on the right. Show the transition to the continuum.

EXERCISE 3.2 A string of length l is released from a position shown in Fig. 3.1. Determine its motion.

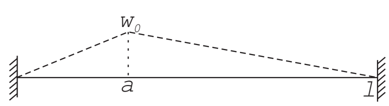

Fig. 3.1 Initial position of string.

Solution. The initial conditions of the spring are

$$w(x,0) = w_0(x) = \begin{cases} \frac{w_0}{a}x & \text{for } x < a, \\ -\frac{w_0}{l-a}(x-a) + w_0 & \text{otherwise,} \end{cases} \qquad w_{,t}(x,0) = 0.$$

The solution to the equation of motion $w_{,tt} = c^2 w_{,xx}$ reads

$$w(x,t) = \sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} \sin\frac{j\pi}{l}x(a_j \cos\omega_j t + b_j \sin\omega_j t),$$

where $\omega_j = j\frac{\pi c}{l}$. The initial conditions yield

$$\sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} a_j \sin\frac{j\pi}{l}x = w_0(x),$$

$$\sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} \omega_j b_j \sin\frac{j\pi}{l}x = v_0(x) = 0.$$

Thus, the coefficients $b_j = 0$. To determine the coefficients a_j we use the orthogonality and normalization condition to get

$$
\begin{aligned}
a_j &= \sqrt{\frac{2}{l}} \int_0^l w_0(x) \sin \frac{j\pi}{l} x\, dx \\
&= \sqrt{\frac{2}{l}} \left[\int_0^a \frac{w_0}{a} x \sin \frac{j\pi}{l} x\, dx + \int_a^l (-\frac{w_0}{l-a}(x-a) + w_0) \sin \frac{j\pi}{l} x\, dx \right] \\
&= \sqrt{\frac{2}{l}} \left[\frac{l w_0 \left(l \sin\left(\frac{\pi a j}{l}\right) - \pi a j \cos\left(\frac{\pi a j}{l}\right)\right)}{\pi^2 a j^2} \right. \\
&\quad \left. + \frac{l w_0 \left(l \left(\sin(\pi j) - \sin\left(\frac{\pi a j}{l}\right)\right) + \pi j (a-l) \cos\left(\frac{\pi a j}{l}\right)\right)}{\pi^2 j^2 (a-l)} \right] \\
&= \sqrt{\frac{2}{l}} \frac{l^3 w_0 \sin\left(\frac{\pi a j}{l}\right)}{\pi^2 j^2 a (l-a)}.
\end{aligned}
$$

Finally, the solution takes the form

$$
w(x,t) = \sum_{j=1}^{\infty} \frac{2 l^2 w_0 \sin\left(\frac{\pi a j}{l}\right)}{\pi^2 j^2 a (l-a)} \sin \frac{j\pi}{l} x \cos j \frac{\pi c}{l} t.
$$

EXERCISE 3.3 An elastic bar of length l has its free end stretched uniformly so that its length becomes $l + u_0$, and then is released from that position (see Fig. 3.2). Determine its motion.

Fig. 3.2 Uniformly stretched bar.

Solution. Let $u(x,t)$ be the longitudinal displacement of the bar. The initial conditions of the bar are

$$
u(x,0) = u_0(x) = \frac{u_0}{l} x, \quad u_{,t}(x,0) = 0.
$$

The solution to the equation of motion $u_{,tt} = c^2 u_{,xx}$ reads

$$
u(x,t) = \sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} \sin \frac{j\pi}{l} x (a_j \cos \omega_j t + b_j \sin \omega_j t),
$$

where $\omega_j = j \frac{\pi c}{l}$. The initial conditions yield

$$\sqrt{\frac{2}{l}}\sum_{j=1}^{\infty} a_j \sin\frac{j\pi}{l}x = u_0(x),$$

$$\sqrt{\frac{2}{l}}\sum_{j=1}^{\infty} \omega_j b_j \sin\frac{j\pi}{l}x = v_0(x) = 0.$$

Thus, the coefficients $b_j = 0$. To determine the coefficients a_j we use the orthogonality and normalization condition to get

$$a_j = \sqrt{\frac{2}{l}}\int_0^l \frac{u_0}{l}x\sin\frac{j\pi}{l}x\,dx = \sqrt{\frac{2}{l}}\frac{lu_0(-1)^{j+1}}{\pi j}.$$

Finally, the solution takes the form

$$u(x,t) = \sum_{j=1}^{\infty}\frac{2u_0(-1)^{j+1}}{\pi j}\sin\frac{j\pi}{l}x\cos j\frac{\pi c}{l}t.$$

EXERCISE 3.4 An elastic shaft having a rigid disk attached at its free end performs torsional vibrations. The disk has a moment of inertia J_D (see Fig. 3.3). Derive the equation of small vibrations and the boundary conditions from Hamilton's variational principle. Determine the eigenfrequencies.

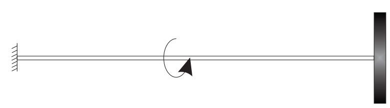

Fig. 3.3 Shaft with rigid disk attached at its end.

Solution. We write down the action functional of this system

$$I[\varphi(x,t)] = \int_{t_0}^{t_1}\int_0^l (\frac{1}{2}\rho J_p\varphi_{,t}^2 - \frac{1}{2}GJ_p\varphi_{,x}^2)\,dx\,dt + \int_{t_0}^{t_1}\frac{1}{2}J_D\varphi_{,t}(l,x)^2\,dt.$$

The last term corresponds to the action functional of the disk. Varying this action functional, we have

$$\begin{aligned}\delta I &= \int_{t_0}^{t_1}\int_0^l (\rho J_p\varphi_{,t}\delta\varphi_{,t} - GJ_p\varphi_{,x}\delta\varphi_{,x})\,dx\,dt + \int_{t_0}^{t_1} J_D\varphi_{,t}(l,x)\delta\varphi_{,t}\,dt \\ &= \int_{t_0}^{t_1}\int_0^l (-\rho J_p\varphi_{,tt} + GJ_p\varphi_{,xx})\delta\varphi\,dx\,dt - \int_{t_0}^{t_1}(GJ_p\varphi_{,x} + J_D\varphi_{,tt})\delta\varphi\,dt = 0\end{aligned}$$

Since $\delta\varphi$ can be chosen arbitrarily in the interval $(0,l)$ and at the end point $x = l$, this equation implies that

$$\rho J_p\varphi_{,tt} - GJ_p\varphi_{,xx} = 0 \Rightarrow \varphi_{,tt} - c^2\varphi_{,xx} = 0$$

inside $(0,l)$, with $c^2 = G/\rho$, and

$$GJ_p\varphi_{,x} + J_D\varphi_{,tt} = 0$$

at $x = l$. Together with the boundary condition at $x = 0$

$$\varphi(0,t) = 0,$$

this constitutes the eigenvalue problem. To determine the spectrum of this system we seek for the solution in the form

$$\varphi(x,t) = q(x)e^{i\omega t}.$$

Substituting into the equation of motion and the boundary condition we obtain

$$\omega^2 q + c^2 q'' = 0,$$

and

$$q(0) = 0, \quad GJ_p q'(0) - J_D\omega^2 q(0) = 0.$$

From the equation for $q(x)$ we find that

$$q(x) = A\cos\frac{\omega}{c}x + B\sin\frac{\omega}{c}x$$

The boundary condition $q(0) = 0$ yields $A = 0$. The other boundary condition at $x = l$ leads to the transcendental equation

$$GJ_p\frac{\omega}{c}\cos\frac{\omega}{c}l - J_D\omega^2\sin\frac{\omega}{c}l = 0,$$

or

$$\tan\frac{\omega}{c}l = \frac{GJ_p}{J_D}\frac{1}{\omega c}.$$

EXERCISE 3.5 Find the eigenfrequencies of flexural vibrations of a beam with one clamped edge and one free edge. Plot the shapes of first three modes of vibrations.

EXERCISE 3.6 The beam of length l and mass m sketched in Fig. 3.4 is released and latches upon impact onto the support B. Provided there is no rebound and no loss of energy, determine the flexural vibration of the beam after impact. How to proceed if there is a rebound.

Solution. Before impact the beam experiences a free falling. The conservation of energy yields

$$\frac{1}{2}J_A\dot{\varphi}_0^2 = mg\frac{h}{2},$$

where $\dot{\varphi}_0$ is the angular velocity of the beam immediately before impact, and

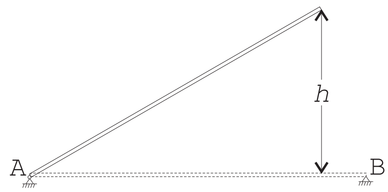

Fig. 3.4 Falling beam.

$$J_A = J_S + m(l/2)^2 = m\frac{l^2}{12} + m\frac{l^2}{4} = m\frac{l^2}{3}$$

is the moment of inertia of the beam abot A. Thus, the angular velocity $\dot{\varphi}_0$ is equal to

$$\dot{\varphi}_0 = \sqrt{\frac{3gh}{l^2}}.$$

Knowing this angular velocity before impact, we find the initial conditions of the beam

$$w(x,0) = w_0(x) = 0, \quad w_{,t}(x,0) = \dot{\varphi}_0 x.$$

The solution to the equation of motion $\mu w_{,tt} = EIw_{,xxxx}$ reads

$$u(x,t) = \sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} \sin\frac{j\pi}{l}x(a_j \cos\omega_j t + b_j \sin\omega_j t),$$

where $\omega_j = (j\pi)^2\sqrt{\frac{EI}{\mu l^4}}$. The initial conditions yield

$$\sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} a_j \sin\frac{j\pi}{l}x = 0,$$

$$\sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} \omega_j b_j \sin\frac{j\pi}{l}x = v_0(x) = \dot{\varphi}_0 x.$$

Thus, the coefficients $a_j = 0$. To determine the coefficients b_j we use the orthogonality and normalization condition to get

$$b_j = \frac{1}{\omega_j}\sqrt{\frac{2}{l}} \int_0^l \dot{\varphi}_0 x \sin\frac{j\pi}{l}x\,dx = \frac{1}{\omega_j}\sqrt{\frac{2}{l}}\frac{l^2(-1)^{j+1}}{\pi j}.$$

Finally, the solution takes the form

$$u(x,t) = \sum_{j=1}^{\infty} \frac{2\dot{\varphi}_0 l(-1)^{j+1}}{\omega_j \pi j} \sin\frac{j\pi}{l}x \sin\omega_j t.$$

EXERCISE 3.7 Derive the boundary condition for a beam connected with a spring shown in Fig. 3.5. Find the eigenfrequencies.

Fig. 3.5 Beam with spring.

EXERCISE 3.8 An elastic beam is subjected to a harmonic end load as shown in Fig. 3.6. Determine its forced vibration.

Fig. 3.6 Beam under harmonic end load.

Solution. The vibration of the beam must be the extremal of the following action functional

$$I[w(x,t)] = \int_{t_0}^{t_1}\int_0^l [\frac{1}{2}\mu w_{,t}^2 - \frac{1}{2}EI(w_{,xx})^2]\,dx\,dt + \int_{t_0}^{t_1} f(t)w(l,t)\,dt,$$

where the last term describes the virtual work done by the concentrated load. Varying this action functional we have

$$\begin{aligned}\delta I &= \int_{t_0}^{t_1}\int_0^l (\mu w_{,t}\delta w_{,t} - EIw_{,xx}\delta w_{,xx})\,dx\,dt + \int_{t_0}^{t_1} f(t)\delta w(l,t)\,dt \\ &= \int_{t_0}^{t_1}\int_0^l (-\mu w_{,tt} - EIw_{,xxxx})\delta w\,dx\,dt \\ &- \int_{t_0}^{t_1} EIw_{,xx}\delta w_{,x}(l,t)\,dt + \int_{t_0}^{t_1}(EIw_{,xxx} + f(t))\delta w(l,t)\,dt = 0.\end{aligned}$$

This implies the equation of motion

$$\mu w_{,tt} + EIw_{,xxxx} = 0,$$

and the boundary conditions at $x = l$

$$w_{,xx} = 0, \quad EIw_{,xxx} + f(t) = 0.$$

Together with the kinematic boundary condition at $x = 0$

$$w(0,t) = 0, \quad w_{,x}(0,t) = 0,$$

this constitutes the boundary-value problem to determine the forced vibration. For the harmonic end load $f(t) = -\hat{f}\cos\omega t$ we look for the solution in the form

$$w(x,t) = q(x)\cos\omega t.$$

Substituting into the equation of motion and the boundary condition we obtain

$$q'''' - \kappa^4 q = 0,$$

with $\kappa^4 = \omega^2\mu/EI$, and

$$q(0) = 0, \quad q'(0) = 0,$$

as well as

$$q''(l) = 0, \quad q'''(l) = \frac{\hat{f}}{EI}.$$

Thus, the solution reads

$$q(x) = C_1 \sin\kappa x + C_2 \cos\kappa x + C_3 \sinh\kappa x + C_4 \cosh\kappa x.$$

Substituting this solution into the above boundary conditions we get four linear equations to determine four coefficients C_1, C_2, C_3, C_4

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin\lambda & -\cos\lambda & \sinh\lambda & \cosh\lambda \\ -\cos\lambda & \sin\lambda & \cosh\lambda & \sinh\lambda \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\hat{f}}{EI\kappa^3} \end{pmatrix},$$

where $\lambda = \kappa l$.

EXERCISE 3.9 A square membrane is subjected to a harmonic load acting at its center. Determine the forced vibration.

EXERCISE 3.10 Determine the eigenfrequencies of a circular plate with a simply supported boundary.

EXERCISE 3.11 Prove the extremal properties of eigenfrequencies of a continuous oscillator based on the minimization of Rayleigh's quotient.

EXERCISE 3.12 Find the spectrum of radial vibrations for an elastic isotropic sphere of radius a.

Chapter 4
Linear waves

EXERCISE 4.1 Solve the 1-D wave equation with $c = 1$ and with the following initial conditions

$$u(x,0) = 0, \quad u_{,t}(x,0) = \begin{cases} x+1 & \text{for } x \in (-1,0), \\ 1-x & \text{for } x \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

Plot the solution at $t = 0.5$ and at $t = 10$.

EXERCISE 4.2 For waves propagating in an infinite elastic material which is homogeneous and isotropic we seek particular solutions in form of plane waves $\mathbf{u} = \mathbf{a}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. Show that there are two velocities of propagation given by

$$c_d = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_s = \sqrt{\frac{\mu}{\rho}},$$

corresponding to dilatational waves (**a** is parallel to **k**) and shear waves (**a** is orthogonal to **k**). Generalize this to homogeneous anisotropic materials.

EXERCISE 4.3 Consider the "balloon problem" in acoustics: the pressure inside a sphere of radius R_0 is $p_0 + P$ while the pressure outside is p_0. The gas is initially at rest, and the balloon is burst at $t = 0$. The initial conditions for the velocity potential reads

$$\varphi(\mathbf{x},0) = 0, \quad \varphi_{,t}(\mathbf{x},0) = \begin{cases} -P/\rho_0 & r < R_0, \\ 0 & \text{otherwise.} \end{cases}$$

Find the change of pressure with time.

EXERCISE 4.4 Search for particular solution in form of plane waves and derive the dispersion relation for 1-D waves propagating in Timoshenko's beam, the dimensionless Lagrangian of which is

$$L=\frac{1}{2}(w_{,t}^2+\alpha u_{,t}^2)-\frac{1}{2}[su_{,x}^2+\beta^2\alpha(u+w_{,x})^2].$$

Plot the dispersion curves and study their asymptotic behavior as $k\to 0$ and $k\to\infty$.

EXERCISE 4.5 Solve the linearized Korteweg-de Vries equation with $\alpha=0$, $\beta=1$ and with the initial condition $u(x,0)=e^{-x^2}$. Compute Fourier's integral numerically[1] and plot the solution at $t=100$.

EXERCISE 4.6 Use the method of stationary phase to find the asymptotically leading term of the solution obtained in the previous exercise as $t\to\infty$ at fixed x/t. Compare this asymptotic solution with the exact one.

EXERCISE 4.7 Show that the lowest branches of the dispersion curves of F- and L-waves in an elastic waveguide approach the straight line $\omega=v_r k$ as $k\to\infty$.

EXERCISE 4.8 Prove that all high-frequency thickness branches of F- and L-waves in an elastic waveguide approach the line $\omega=k$ from above as $k\to\infty$.

EXERCISE 4.9 Derive the following asymptotic formulas valid in the long-wave range

$$\omega^2=\omega_c^2+\left(\frac{1}{\eta^2}-\frac{16\tan(\omega_c/2)}{\omega_c}\right)k^2,$$

where $\omega_c=2\pi n/\eta$, for the branch $F_\perp$(n), and

$$\omega^2=\omega_c^2+\left(1+\frac{16\eta\cot(\eta\,\omega_c/2)}{\omega_c}\right)k^2,$$

where $\omega_c=\pi(2n+1)$, for the branch $F_\parallel$(n) of the flexural waves in an elastic waveguide.

EXERCISE 4.10 Derive the equation of energy propagation for Timoshenko's beam using the variational-asymptotic method and compare it with the similar equation obtained from averaging the energy balance equation.

EXERCISE 4.11 Solve the strip problem for 3-D Klein-Gordon equation to find the average Lagrangian, the dispersion relation, and the equation of energy propagation.

EXERCISE 4.12 Derive the following equations

$$(\omega\bar{L}_{,\omega}-\bar{L})_{,t}+(-\omega\bar{L}_{,k_\alpha})_{,\alpha}=0,$$
$$(k_\alpha\bar{L})_{,t}+(-k_\alpha\bar{L}_{,k_\beta}+\bar{L}\delta_{\alpha\beta})_{,\beta}=0,$$

[1] Since the integrand is highly oscillatory, the accuracy is achieved only by increasing the maximum number of recursive subdivisions.

for homogeneous media, which can be interpreted as the energy and “wave momentum” equations, respectively. What happens if $\bar{L}$ depends on the slow variables x_α and t.

Chapter 5
Autonomous single oscillator

EXERCISE 5.1 A point-mass m moves under the action of gravity along a frictionless circular wire of radius r that is rotating with a constant angular velocity Ω about its vertical diameter (see Fig. 5.1)[1]. Derive the equation of motion. Plot the potential energy and the phase portrait.

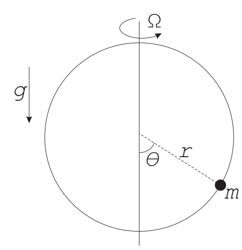

Fig. 5.1 Point-mass on rotating circular wire.

EXERCISE 5.2 Do the next step of the variational-asymptotic procedure for Duffing's equation and show that

$$T = 2\pi\left[1 - \varepsilon\frac{3}{8}a^2 + \varepsilon^2\frac{57}{256}a^4 + O(\varepsilon^3)\right].$$

EXERCISE 5.3 Consider a mass-spring oscillator with an asymmetric spring obeying the equation

$$\ddot{x} + x + \varepsilon x^2 = 0.$$

Find the period of vibration for $\varepsilon = 0.1$ and $x(0) = 1$, $\dot{x}(0) = 0$ using the numerical integration based on (5.3). Compare it with the result obtained by the variational-asymptotic method.

[1] A pendulum oscillating on a rotating platform can serve as a similar example.

EXERCISE 5.4 Find and classify the fixed points of equation (5.14) of a damped pendulum for all $c > 0$, and plot the phase portraits for the qualitatively different cases.

EXERCISE 5.5 The motion of a mass-spring oscillator with the linear restoring force $-kx$ ($k = 2$N/cm) is damped by a constant braking force $f_r = 1$N, this force acts however only in the region $-1\text{cm} \le x \le 1\text{cm}$. Outside this region the oscillator carries out a free vibration. Find the sequence of turning points and the number of halves of vibrations for the initial conditions $x = -3$cm and $\dot{x} = 0$.

EXERCISE 5.6 Consider a damped pendulum with "turbulent" damping described by the equation

$$\ddot{\varphi} + c\dot{\varphi}|\dot{\varphi}| + \omega_0^2 \sin\varphi = 0.$$

Find the sequence of turning angles.

EXERCISE 5.7 Consider Froude's pendulum described by the following dimensionless equation

$$\ddot{\varphi} + 2\delta\dot{\varphi} + \omega_0^2 \sin\varphi = m_r(\dot{\varphi} - \nu_0),$$

where

$$2\delta = \frac{c}{J}, \quad \omega_0^2 = \frac{mgl}{J}, \quad m_r = \frac{M_r}{J}.$$

Find conditions, under which this oscillator develops self-sustained vibrations.

EXERCISE 5.8 Consider the mechanical system governed by the differential equation

$$\ddot{x} + \varepsilon \sin\dot{x} + x = 0.$$

Construct several phase curves for $\varepsilon = 0.1$ using numerical integration. Show that more than one limit cycle exists. Use the variational-asymptotic method to calculate the amplitude of limit cycles.

EXERCISE 5.9 Show that Rayleigh's equation

$$\ddot{x} + \dot{x} - \varepsilon(1 - \frac{1}{3}\dot{x}^2)\dot{x} = 0$$

can be written as van der Pol's equation

$$\ddot{u} + \dot{u} - \varepsilon(1 - u^2)\dot{u} = 0,$$

where $u = \dot{x}$. Find its limit cycle for small ε.

EXERCISE 5.10 Consider the equation

$$\ddot{x} + \dot{x} + \mu(|x| - 1)\dot{x} = 0.$$

Find the approximate period and amplitude of the limit cycle for small and large μ.

EXERCISE 5.11 Use the variational-asymptotic method to study the equation

$$\ddot{x} + \dot{x} - \varepsilon(1 - x^4)\dot{x} = 0$$

for small ε. Find the approximate amplitude of the limit cycle.

EXERCISE 5.12 Use the variational-asymptotic method to study the equation

$$\ddot{x} + \dot{x} - \mu(1 + x - x^2)\dot{x} = 0,$$

where μ is a large parameter. Find the amplitude and period of the limit cycle. Compare the results with those obtained by numerical integration for $\mu = 10$.

Chapter 6
Non-autonomous single oscillator

EXERCISE 6.1 A point-mass m is constrained to move in the (x,y)-plane and is restrained by two linear springs of equal stiffness k and equal unstretched length l. The anchor points of the springs are located on the x-axis at $x=-b$ and $x=b$ (see Fig. 6.1). Study the stability of the motion along the x-axis, $x = a\cos\omega_0 t$, $y=0$ under the assumption that $a \ll b$.

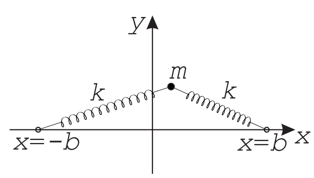

Fig. 6.1 Point-mass in (x,y)-plane.

EXERCISE 6.2 The support of a pendulum considered in example 6.1 moves in accordance with the equation $x = a_0\cos\omega t$, where $a_0 = 0.1l$. How large must the frequency ω be to stabilize the vertical position $\varphi = \pi$.

EXERCISE 6.3 Apply the variational-asymptotic method to find the asymptotes of the transition curves of Mathiew's equation emanating from the point $\mu = 1$.

EXERCISE 6.4 Consider the damped Mathiew equation

$$\ddot{x} + \varepsilon c\dot{x} + (\mu + \varepsilon\cos t)x = 0,$$

with ε being a small parameter. Apply the variational-asymptotic method to find the asymptotes of the transition curves near the point $\mu = 1/4$.

EXERCISE 6.5 Non-linear parametric resonance. Consider the following equation

$$\ddot{x} + \omega_0^2 x + \varepsilon \cos t \, x^3 = 0,$$

with ε being a small parameter. Apply the variational-asymptotic method to study the behavior of solutions near the frequency $\omega_0 = 1/2$.

EXERCISE 6.6 Solve the slow flow system (6.26) numerically for $\varepsilon = 0.1$, $c = 0$, $\alpha = \hat{f} = 1$ and for two detuning values $k_1 = 0$ and $k_1 = -0.125$, with the initial conditions $A(0) = 1$ and $B(0) = 0$. Plot the curves $a(\tau) = \sqrt{A^2 + B^2}$ together with the numerical solutions shown in Figs. 6.8 and 6.9.

EXERCISE 6.7 Find the steady-state amplitude versus frequency curve of the forced Duffing equation with the softening spring ($\alpha < 0$). Discuss the jump phenomenon and the hysteresis loop.

EXERCISE 6.8 Consider the forced oscillator with the quadratic damping described by the equation

$$\ddot{x} + x + \varepsilon c \dot{x} |\dot{x}| = \varepsilon \hat{f} \cos \omega t,$$

where ε is small. Apply the variational-asymptotic method to find the amplitude versus frequency curve near the 1:1 resonant frequency.

EXERCISE 6.9 Consider the forced Duffing oscillator described by the equation

$$\ddot{x} + x + \varepsilon c \dot{x} + \varepsilon \alpha x^3 = \hat{f} \cos \omega t,$$

where ε is small, but $\hat{f}$ is finite (sometimes called a “hard excitation”). Apply the variational-asymptotic method to show that to $O(\varepsilon)$, the only resonant frequencies are 1,3, and 1/3.

EXERCISE 6.10 Study the excitation of 3:1 subharmonics in the previous exercise by setting $\omega = 3 + k_1 \varepsilon$. Obtain a slow flow of the coefficients $A(\eta)$ and $B(\eta)$. Then transform to the polar coordinates $a(\eta)$ and $\psi(\eta)$ and look for fixed points of those equations. Eliminate ψ in order to find a relation between a^2 and other parameters. For $\alpha = c = \hat{f} = 1$ plot a versus k_1.

EXERCISE 6.11 Solve the slow flow system (6.36) numerically for $\varepsilon = 0.1$, $k_1 = 0.2$ and $k_1 = 0.5$, with the initial conditions $a(0) = 1$ and $\psi(0) = 0$. Plot the curves $a(\tau)$ together with the numerical solutions shown in Figs. 6.16 and 6.17.

EXERCISE 6.12 Subharmonic resonance. Consider the forced van der Pol’s oscillator described by the equation

$$\ddot{x} + x - \varepsilon (1 - x^2) \dot{x} = \hat{f} \cos \omega t,$$

where ε is small, but $\hat{f}$ is finite. Apply the variational-asymptotic method to show that to $O(\varepsilon)$, the only resonant frequencies are 1,3, and 1/3. Study the subharmonic 3:1 resonance case.

Chapter 7
Coupled oscillators

EXERCISE 7.1 Derive the equations of nonlinear vibration of the double pendulum considered in exercise 2.1.

EXERCISE 7.2 Hamilton-Jacobi equation. Let the action function $S(\mathbf{q},t)$ be defined as the integral

$$S_{\mathbf{q}_0,t_0}(\mathbf{q},t) = \int_\gamma L\,dt$$

along the extremal γ connecting the points $(\mathbf{q}_0,t_0)$ and $(\mathbf{q},t)$. Show that $S(\mathbf{q},t)$ satisfy the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}) = 0.$$

EXERCISE 7.3 Find the action variable for the Duffing oscillator with

$$H(q,p) = \frac{1}{2}(p^2 + U(q)), \quad U = U(q) = \frac{1}{2}q^2 + \frac{1}{4}\alpha q^4.$$

EXERCISE 7.4 Simulate numerically the Poincaré map for the Hénon-Heiles equations which can be obtained as Lagrange's equation of the following Lagrange function

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(x^2 + y^2 + 2x^2 y - \frac{2}{3}y^3).$$

Choose the cut plane $x = 0$ and the total energy i) $E_0 = 0.01$ and ii) $E_0 = 1/8$. Observe the difference in cases i) and ii).

EXERCISE 7.5 Modal equation in a rotating frame. In the frame rotating with the constant angular velocity ω, the presence of Coriolis and centripetal accelerations changes the equations of motion (7.7) to

$$\ddot{x} - 2\omega\dot{y} - \omega^2 x = -\frac{\partial U}{\partial x}, \quad \ddot{y} + 2\omega\dot{x} - \omega^2 y = -\frac{\partial U}{\partial y}$$

For this system, obtain a first integral and using it to derive a modal equation for the orbits in the (x, y)-plane which does not involve time t.

EXERCISE 7.6 Derive equations (7.14).

EXERCISE 7.7 Compute the approximate Poincaré map from the first integral (7.17) numerically for the energy level $E_0 = 0.4$ and for the parameter $\varepsilon = 0.1$, $\kappa = 0.4$, and compare it with the Poincaré map obtained by the numerical integration of the exact equations.

EXERCISE 7.8 Show that the set of strongly resonant frequencies satisfying (7.20) is not empty and has the full Lebesgue measure if $\nu > n - 1$.

EXERCISE 7.9 Prove the formulas $(7.29)_{2,3}$.

EXERCISE 7.10 Simulate numerically the solutions of equations (7.30) satisfying the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$ and $y(0) = 1$, $\dot{y}(0) = 0$ for $\varepsilon = 0.1$, $\alpha = 1$, and $\kappa = 1.2$. Plot the curves $x(t)$, $y(t)$, and $x(t)y(t)$. Explain why synchronization leads to the stationary behavior of the amplitude modulation of $x(t)y(t)$.

EXERCISE 7.11 Recheck the slow flow equations (7.33)

EXERCISE 7.12 Solve the slow flow system (7.33) numerically for $\alpha = 1$, and $\kappa = 1.2$, with the initial conditions $a_1(0) = 1$, $a_2(0) = 1$, and $\varphi(0) = 1$. Plot the curves $a_1(\eta)$, $a_2(\eta)$, and $\varphi(\eta)$ and observe their behavior as η becomes large.

Chapter 8
Nonlinear waves

EXERCISE 8.1 Use the identities for the Jacobian elliptic functions sn, cn, and dn given in Section 5.1 to check that $\varphi(\xi) = a\,\mathrm{cn}^2(\sqrt{b/2}\xi, a/b)$, with $\xi = x - ct$, is the periodic solution of the KdV equation (in this case $b_1 = a - b$, $b_2 = 0$, $b_3 = a$).

EXERCISE 8.2 Show that

$$u(x,t) = 4\arctan e^{\gamma(x-ct)},$$

with $\gamma = 1/\sqrt{1-c^2}$ is the soliton solution of the Sine-Gordon equation.

EXERCISE 8.3 Use the conservation law of the KdV equation

$$u_{,t} + (3u^2 + u_{,xx})_{,x} = 0$$

to show that

$$I_{-1} = \int_{-\infty}^{\infty} u\,dx$$

is the first integral. Show that the conservation laws of the KdV equation for I_0 and I_1 are

$$(u^2)_{,t} + (4u^3 + 2uu_{,xx} - u_{,x}^2)_{,x} = 0,$$
$$(u^3 - \frac{1}{2}u_{,x}^2)_{,t} + (\frac{9}{4}u^4 + 3u^2u_{,xx} - 6uu_{,x}^2 - u_{,x}u_{,xxx} + \frac{1}{2}u_{,xx}^2)_{,x} = 0.$$

EXERCISE 8.4 With the Lax pair

$$L\psi = \psi_{,xx} + u(x,t)\psi, \quad A\psi = (\gamma + u_{,x})\psi - (4\lambda + 2u)\psi_{,x},$$

show that the Lax equation $L_{,t} + [L,A] = 0$ together with $L\psi = \lambda\psi$ is compatible with the KdV equation.

EXERCISE 8.5 Consider two linear equations

$$\mathbf{v}_{,x} = \mathbf{X}\mathbf{v}, \quad \mathbf{v}_{,t} = \mathbf{T}\mathbf{v},$$

where $\mathbf{v}$ is an n-dimensional vector and $\mathbf{X}$ and $\mathbf{T}$ are $n \times n$ matrices. Provided these equations are compatible, that is $\mathbf{v}_{,xt} = \mathbf{v}_{,tx}$, show that $\mathbf{X}$ and $\mathbf{T}$ satisfy

$$\mathbf{X}_{,t} - \mathbf{T}_{,x} + [\mathbf{X}, \mathbf{T}] = 0.$$

The pair $\mathbf{X}$ and $\mathbf{T}$ is similar to Lax's pair L and A, and the last equation may lead to various interesting equations of mathematical physics.

EXERCISE 8.6 Consider the two-soliton solution

$$u(x,t) = 12 \frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}.$$

Plot this function for the time instants before, during, and after the collision. Observe the behavior of the amplitudes and phases.

EXERCISE 8.7 Find the average Lagrangian by solving the minimization problem

$$\bar{L} = \frac{1}{2\pi} \min_{\psi_1, \psi_2} \int_0^{2\pi} [\frac{1}{2}(\omega^2 - k^2)(\psi_{1,\theta}^2 + \psi_{2,\theta}^2) - U(\psi_1, \psi_2)]\, d\theta,$$

where

$$U(\psi_1, \psi_2) = \frac{1}{2}[\psi_1^2 + \frac{\alpha}{2}\psi_1^4 + \psi_2^2 + \frac{\alpha}{2}\psi_2^4 + \frac{\beta}{2}(\psi_2 - \psi_1)^4],$$

among 2π-periodic functions for which $\psi_2 = c\psi_1$.

EXERCISE 8.8 For the average Lagrange function

$$\bar{L} = \frac{\omega}{2\pi} \int_0^T p\dot{q}\, dt - h = \frac{\omega}{2\pi} \oint p(q, h, \lambda)\, dq - h,$$

of an oscillator depending on the slowly changing parameter λ show that $\partial\bar{L}/\partial h = 0$ coincides with the amplitude-frequency equation.

EXERCISE 8.9 Derive equation (8.31).

EXERCISE 8.10 Transform equations (8.41) to (8.42), (8.43) and their cyclic permutations.

EXERCISE 8.11 Find the average Lagrangian (8.52).

EXERCISE 8.12 Assume the initial condition of the KdV equation as $u(x,0) = u_0(x)$, where $u_0(x)$ is a rectangular function of width l and height A. Find the asymptotic solution by the inverse scattering transform for the case of large $S = l\sqrt{A}$. Compare this solution with (8.53).